4d-flat compactifications with brane vorticities

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Abstract

We present solutions in six-dimensional gravity coupled to a sigma model, in the presence of three-brane sources. The space transverse to the branes is a compact non-singular manifold. The example of $O(3)$ sigma model in the presence of two three-branes is worked out in detail. We show that the four-dimensional flatness is obtained with a single condition involving the brane tensions, which are in general different and may be both positive, and another characteristic of the branes, vorticity. We speculate that the adjustment of the effective four-dimensional cosmological constant may occur through the exchange of vorticity between the branes. We then give exact instanton type solutions for sigma models targeted on a general Kähler manifold, and elaborate in this framework on multi-instantons of the $O(3)$ sigma model. The latter have branes, possibly with vorticities, at the instanton positions, thus generalizing our two-brane solution.

Theories with extra dimensions offer interesting twists of the cosmological constant problem [1]. In brane-world models, four-dimensional flatness in general does not require that the tension of the Standard Model brane vanishes. The effect of this tension on four-geometry may be compensated by the bulk cosmological constant and/or bulk fields, as well as by tensions of other branes. It is difficult, however, to invent an adjustment mechanism for the effective four-dimensional cosmological constant in those cases, since the parameters balancing the SM brane tension either are constants of motion or depend on properties of other branes.

In this paper we present a model of somewhat different sort, with two compact non-singular extra dimensions and several branes whose tensions are in general different and not tuned. Besides the six-dimensional Einstein gravity, our model involves a scalar field of a non-linear sigma model. Thus, our solutions[3] are brane-world generalisations of the solution found in Ref. [2]. We begin with a simple example of an $O(3)$ sigma model and a solution which generalizes the flat-space one-instanton solution of Ref. [5]. In this case the topology of the transverse space is that of S^2, and there are two branes placed at the poles. A novel property is the vorticity

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[3]Singular solutions of this type in supergravity models have been found in [3]. Solutions from the same class as ours have been found independently in Ref. [4], which appeared after the arXive version of this paper. The authors of Ref. [4] do not, however, introduce vorticities.

of a brane, which may be thought of as the Aharonov–Bohm phase of the scalar field around the brane. In compact transverse space, the vorticities of the two branes are necessarily equal and opposite. Four-dimensional flatness requires one relation between the brane tensions and vorticities. Since the overall vorticity is zero, it is not inconceivable that the vorticity of each brane may vary in time due to the vorticity exchange. It is thus tempting to speculate that the adjustment of the effective four-dimensional cosmological constant may occur through the exhange of vorticity between the branes.

In our model both brane tensions may be positive. The compact transverse space has the shape of distorted two-sphere with conical defects at the two poles. Geometries of this type have been encountered in previous attempts to sidestep the cosmological constant problem [6, 7] (see also Refs. [8, 9, 10]). The model which has been studied by these authors consisted of the six-dimensional Einstein–Maxwell system with a bulk cosmological constant. It was shown long time ago that $R^4 \times S^2$ with the Maxwell field assuming magnetic monopole configuration is a stable solution of this system [12]. The authors of Refs. [6, 7] have shown that the same topology continues to be a solution in the presence of delta-function type three-branes too. However, to obtain this, a certain relationship between the brane tensions and the parameters of the action has been required [13], which is not the case in our model.[4]

We then proceed to general sigma models targeted on arbitrary Kähler manifolds. Remarkably, we are able to present a class of solutions to the system of the scalar field and Einstein equations in a fairly concrete form. Making use of this result, we give explicit multi-instanton solutions of $O(3)$ sigma model coupled to gravity. Like in the flat case, the instanton positions are moduli of the soultions. These multi-instantons have branes, possibly with vorticities, at their centers, thus generalizing our two-brane solution.

We start from the action

$$S = \int \sqrt{-G}\left[M^{-4}R - \frac{1}{2\lambda^2}\nabla^M\phi^\alpha\nabla_M\phi^\beta h_{\alpha\beta}(\phi) + L_{brane}\right]$$

Here M is the six-dimensional Planck mass and $\phi^\alpha(x)$, in the general case, are real scalar fields parameterizing a Kähler manifold with the metric $h_{\alpha\beta}$. For the example of the $O(3)$ model the target space is the sphere S^2 with the metric $h_{\alpha\beta}$ given by

$$h_{\alpha\beta} = \frac{4}{(1+\frac{\phi^2}{\alpha^2})^2}\delta_{\alpha\beta} \tag{1}$$

where $\phi^2 = \phi_1^2 + \phi_2^2$ and α is the radius of S^2. One can absorb this parameter into a redefinition of λ; henceforth we set $\alpha = 1$. The six-dimensional Einstein equations are

$$R_{MN} - \frac{1}{2}G_{MN}R = M^{-4}T_{MN}$$

where the energy-momentum tensor is given by

$$T_{MN} = \frac{2}{\lambda^2}h_{\alpha\beta}(\nabla_M\phi^\alpha\nabla_N\phi^\beta - \frac{1}{2}G_{MN}\nabla^L\phi^\alpha\nabla_L\phi^\beta) + T_{MN}^{brane}$$

The brane energy-momentum tensor will be taken to represent a pair of three-branes parallel to each other. Note that we set the bulk cosmological constant equal to zero; we shall comment on this point later on.

The scalar fields satisfy the highly non-linear equations

$$\nabla^M\nabla_M\phi^\alpha + \Gamma^\alpha_{\beta\gamma}(\phi)\nabla^L\phi^\beta\nabla_L\phi^\gamma = 0 \tag{2}$$

[4] For a detailed study of such models see Ref. [11].

where Γ's are the connection components in the space of the ϕ's. Our ansatz for the solution is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \psi(y)\delta_{mn} dy^m dy^n$$

where $\mu, \nu = 0, 1, 2, 3$ and y^m are the local Gaussian coordinates in the two dimensional manifold which will support the branes whose world volumes are along the x^μ subspace. $g^{(2)}_{mn}(y) = \psi(y)\delta_{mn}$ represent the components of the metric in the space transverse to the branes.

The equations given above are general. Let us consider first the example of $O(3)$ sigma model. The simplest anzatz for the scalar fields is

$$\phi^\alpha = y^\alpha \tag{3}$$

It is straightforward to verify that this ansatz solves the scalar field equations (2) with no constraints on the parameters of the model. More general solutions will be given later on.

To obtain the solution to the Einstein equations, we assume that the transverse space is symmetric under $O(2)$ rotations, and write its metric as follows,

$$ds_2^2 = \psi(r)(dr^2 + r^2 d\varphi^2) \tag{4}$$

where ψ is a function of r only. The contribution of the scalar fields of the form (3) to T_{mn} vanishes identically. The only non-vanishing components of T_{MN} then become

$$T_{\mu\nu} = \eta_{\mu\nu} \frac{1}{\psi} \left[-\frac{8}{\lambda^2} \frac{1}{(1+r^2)^2} - \sum T_i \delta_2(y - y_i) \right]$$

The location of the branes with the tension T_i has been denoted by y_i. The only non-trivial information is contained in the $\mu\nu$-components of the Einstein equations, which yield

$$\psi R^{(2)} = \frac{16}{\lambda^2 M^4} \frac{1}{(1+r^2)^2} + \frac{2}{M^4} \sum T_i \delta^{(2)}(y_- y_i) \tag{5}$$

where $R^{(2)} = -\frac{1}{\psi}\Delta^{(2)} \ln \psi$ is the scalar curvature of the two-dimensional transverse space. Outside the branes, the solution to eq. (5) is

$$\psi(r) = \beta^2 \frac{r^{-\frac{\tau_0}{\pi M^4}}}{(1+r^2)^{\frac{4}{\lambda^2 M^4}}} \tag{6}$$

where β and τ_0 are yet undetermined constants. For

$$\frac{\tau_0}{2\pi M^4} + \frac{4}{\lambda^2 M^4} > 1$$

the proper distance from the origin to $r = \infty$, as well as the volume of the transverse space are finite. The transverse space thus has the topology of S^2. In general, there are conical defects at $r = 0$ and $r = \infty$, i.e., there are branes of non-vanishing tensions at the two poles.

We still have to specify the range of the coordinate φ. Let us choose

$$\varphi \in [0, 2\pi(1-\kappa)] \tag{7}$$

where κ is yet another parameter of the model. With this choice, we introduce the vorticities of the branes placed at $r = 0$ and $r = \infty$. Indeed, the scalar field configuration (3) may be written as

$$\phi^1 + i\phi^2 = re^{i\varphi}$$

It is not single valued: as the angle φ makes full rotation, i.e., changes from 0 to $2\pi(1-\kappa)$, the field obtains the phase factor $\exp(-2\pi i\kappa)$. Yet the physical quantities like energy-momentum tensor

or currents are single-valued, so this construction makes sense. Physically, it may be realised, e.g., if the $U(1)$ symmetry $\phi \to \exp(i\alpha)\cdot\phi$ is gauged (with negligibly small gauge coupling), and the brane at the origin carries the Aharonov–Bohm flux proportional to κ. Clearly, the brane at $r=\infty$ has the vorticity of equal magnitude and opposite sign (e.g., it carries the opposite Aharonov–Bohm flux).

Near the origin, a coordinate transformation brings the metric (4) to the metric of the 2-dimensional Euclidean plane,

$$ds^2 = d\rho^2 + \nu^2\rho^2 d\varphi^2$$

where $\nu = 1 - \frac{\tau_0}{2\pi M^4}$. Introducing a new polar angle $\varphi' = \nu\varphi$, we recover the standard flat metric except that φ' ranges from zero to $2\pi\nu(1-\kappa)$. We thus obtain a deficit angle $\delta = 2\pi(1-\nu+\nu\kappa)$. Making use of the standard relation between the brane tension at the origin, T_0, and the deficit angle, namely, $\delta = \frac{T_0}{M^4}$, we express the parameter τ_0 of the solution through the tension and vorticity of the brane placed at the origin,

$$\tau_0 = \frac{T_0 - 2\pi\kappa M^4}{1-\kappa} \tag{8}$$

Another parameter of the solution, β, remains undetermined; it is thus a modulus.

As r approaches infinity, the metric becomes

$$ds_2^2 = \beta^2 \xi^{-\frac{\tau_\infty}{\pi M^4}} (d\xi^2 + \xi^2 d\varphi^2)$$

where $\xi = \frac{1}{r}$ and τ_∞ is defined by

$$\tau_\infty + \tau_0 = 4\pi M^4 \left(1 - \frac{2}{\lambda^2 M^4}\right) \tag{9}$$

The relation between τ_∞ and the brane tension at $r=\infty$ is again given by eq. (8), with τ_∞ and T_∞ substituted for τ_0 and T_0, respectively. Therefore, eq. (9) is in fact the relation between the brane tensions and vorticity, for given parameters of the action,

$$T_\infty + T_0 = 4\pi \left(M^4 - \frac{2(1-\kappa)}{\lambda^2}\right) \tag{10}$$

It is this relation that ensures the absense of singularities and four-dimensional flatness of our solution. As discussed above, for given T_0 and T_∞ it requires the tuning of the vorticity κ.

Several remarks are in order.

(i) Clearly, there is a domain of parameter space where the tensions of both branes can be positive. The vorticity can have either sign.

(ii) As a cross check, one can calculate the Euler number of the transverse space, and find that it is equal to +2. This reiterates that our space is topologically S^2.

(iii) One finds from eqs. (5) and (6) that the Ricci scalar $R^{(2)}$ vanishes at the brane positions. We will make use of this property to construct more general solutions in what follows.

(iv) To obtain our solution, we have assumed that the six-dimensional cosmological constant is zero. It has been shown very recently [4] that in the model under study, the non-vanishing bulk cosmological constant leads to (anti-) de Sitter geometry on the branes, i.e., it gives rise to non-zero effective four-dimensional cosmological constant. One way to ensure that the six dmensional cosmlogical constant vanishes would be to find a six-dimensional supergravity model, such as the anomaly free model of Ref. [14] where there are several scalar fields parameterizing a non-linear sigma model manifold, and where the six-dimensional cosmological constant is zero by supersymmetry. Unfortunately the target spaces of the sigma models in such supergravities

are not of the type we have assumed. Nevertheless holomorphic embedding is possible, although the most straightforward one yields a singular [3] transverse space. [5]

Let us now show that the solution presented above is the simplest among a wider class of solutions in which the target space of the sigma model can be any Kähler manifold. Such manifolds are complex with the metric tensor derivable from a potential $\chi(\phi^a, \phi^{\bar{a}})$, where ϕ^a and $\phi^{\bar{a}}$ are local complex coordinates. The only non-vanishing components of the Hermitean metric are $h_{a\bar{b}} = \partial_a \partial_{\bar{b}} \chi$, where $\partial_a = \frac{\partial}{\partial \phi^a}$ and $\partial_{\bar{a}}$ is the complex conjugate operator. The connection components can be calculated by using the standard formula. The non-vanishing ones are $\Gamma^a_{bc} = h^{a\bar{d}} \partial_b h_{\bar{d}c}$ and their complex conjugates.

It is convenient to introduce complex coordinates in the transverse y-space too. Let us denote them by z and $\bar{z}$. It is straightforward to see that the ϕ-field equations are satisfied if

$$\partial_{\bar{z}} \phi^a = 0 \tag{11}$$

These are instanton configurations. A remarkable fact is that we can solve the Einstein equations for a general Kähler manifold. The solution is

$$\psi(z, \bar{z}) = |g(z)|^2 \exp\left[-\frac{2}{\lambda^2 M^4} \chi(\phi, \bar{\phi})\right] \tag{12}$$

where $g(z)$ is a function of z but not $\bar{z}$. It is worth noting that the Euler number of the transverse space for this solution can be written in a fairly explcit form as well. We obtain, using the Einstein equations,

$$\frac{1}{4\pi} \int d^2x \, \sqrt{g^{(2)}} R^{(2)} = \frac{2}{\pi \lambda^2 M^4} \int dz d\bar{z} \, \partial_z \partial_{\bar{z}} \chi + \frac{1}{2\pi M^4} \sum T_i \tag{13}$$

Here the first integral on the right hand side is the pull back of the Kähler class of the target space of the sigma model.

We leave the analysis of the properties of these general solutions for future work, and here we make use of eqs. (11) and (12) to obtain multi-instanton solutions in $O(3)$ sigma model.

Let us first consider branes without vorticities. In the case of $O(3)$ model the N-instanton configuration is given by

$$\phi = \prod_{k=1}^{N} \frac{z - a_k}{z - b_k} \tag{14}$$

where a_k and b_k are the $2N$ complex moduli of the instantons. The solution is normalized in such a way that it remains finite as $z \to \infty$. In other words, we choose the coordinates z in the transverse space such that there is no brane at $z = \infty$. For $N = 1$ we obtain our previous solution by a simple holomorphic change of the z-coordinate.

The potential for the $O(3)$ model is $\chi(\phi, \bar{\phi}) = 2 \ln(1 + |\phi|^2)$. We use the analogy to the one-instanton case, and determine the function $g(z)$ in eq. (12) by the requirement that the Ricci scalar $R^{(2)}$ vanishes at $z = a_l$ and $z = b_l$. This condition fixes $g(z)$ to be

$$g(z) = \beta \prod_{1}^{N} \frac{(z - a_l)^{-\frac{\tau_l}{2\pi M^4}}}{(z - b_l)^{-\frac{\tau_l'}{2\pi M^4} + \frac{2}{N}}} \tag{15}$$

where β and τ_l and τ_l' are constants. It is straightforward to find the metric near the instanton centers,

$$ds_2^2 \; = \; |z - a_l|^{-\frac{\tau_l}{\pi M^4}} \, dz d\bar{z} \,, \quad z \to a_k \tag{16}$$

$$ds_2^2 \; = \; |z - b_l|^{-\frac{\tilde{\tau}_l}{\pi M^4}} \, dz d\bar{z} \,, \quad z \to b_k \tag{17}$$

[5]After this paper appeared in arXive, it has been shown in Ref. [15] that analogous non-singular positive tension branes also exist in supergravity models.

where

$$\tilde{\tau}_l = -\tau_l' + \frac{4\pi M^4}{N}\left(1 - \frac{2N}{\lambda^2 M^4}\right) \tag{18}$$

There are thus deficit angles at each of the points $z = a_l$ and $z = b_l$, which indicate the presence of $2N$ three-branes sitting at these points. Their tensions are (recall that we consider branes without vorticities)

$$T_l = \tau_l , \quad z = a_l \tag{19}$$

$$\tilde{T}_l = \tilde{\tau}_l , \quad z = b_l \tag{20}$$

In order that the deficit angles and hence the tensions be positive we need to require that both τ_l and $\tilde{\tau}_l$ are positive. This will also ensure that the Ricci scalar $R^{(2)}$ vanishes at the position of the branes. Finally, the transverse space is compact and does not have a conical singularity at $z = \infty$ provided that $g(z) = 1/z^2$ as $z \to \infty$. This implies a sum rule

$$\sum_1^N (\tau_l - \tau_l') = 0 \tag{21}$$

In view of eqs. (18) and (20) this is in fact a sum rule for the brane tensions, whose explicit form will be given below.

Let us now introduce the brane vorticities. We denote the vorticities around a_l and b_l by κ_l and κ_l', respectively. Full rotations around each of the branes should induce the corresponding phases in ϕ, so the global solution for the scalar field, instead of eq. (14), now has the form

$$\phi = \prod_{k=1}^N \frac{(z - a_k)^{1-\kappa_k}}{(z - b_k)^{1+\kappa_k'}} \tag{22}$$

To make contact with the definition (7), we note that near $z = a_k$, the coordinate transformation $y = (z - a_k)^{1-\kappa_k}$ brings the field (22) into the form (3), while the phase of y ranges from zero to $2\pi(1 - \kappa_k)$, in accord with eq. (7). Similar remark applies to branes at sites b_k.

The field ϕ should tend to a constant as $z \to \infty$ (no brane at $z = \infty$), which implies that the sum of vorticities vanishes,

$$\sum_{l=1}^N (\kappa_l + \kappa_l') = 0 \tag{23}$$

Now, the metric is still given by eqs. (12) and (15), and its behaviour near the instanton centers is again determined by eqs. (16) and (17). However, instead of eq. (18) we now have

$$\tilde{\tau}_l = -\tau_l' + \frac{4\pi M^4}{N}\left[1 - \frac{2N}{\lambda^2 M^4}(1 + \kappa_l')\right]$$

The relations (19) and (20) are still valid, so the sum rule (21) gives one relation between the tension and vorticities,

$$\sum_l (T_l + \tilde{T}_l) = 4\pi \left[M^4 - \frac{2}{\lambda^2}\left(N - \sum_l \kappa_l\right)\right]$$

where we used the sum rule (23). This generalizes the relation (10) to the multi-brane case. Making use of eq. (13) one checks that this solution has the Euler number +2.

To end up this note, let us point out a particular case when $a_1 = a_2 = ... = a_N \equiv a$ and $b_1 = b_2 = ... = b_N \equiv b$. In this case we have essentially a single brane at $z = a$ and another one

at $z = b$. By the coordinate transformation of the form $\xi = \frac{z-a}{z-b}$, the formulae given in the last paragraph reduce to the ones very similar to the one instanton example, except that we need to replace r in the ψ-function by r^N, where $r = |\xi|$. The relationship (10) between the tensions is then replaced by

$$T_\infty + T_0 = 4\pi \left(M^4 - \frac{2N(1-\kappa)}{\lambda^2} \right)$$

This shows that even for $\kappa = 0$, at large λ we obtain a dense, albeit discrete, set in the space of tensions, which is parameterized by the instanton number N.

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